2022

MATHEMATICS — HONOURS

Paper : CC-7

(ODE & Multivariate Calculus - I)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$ denotes the set of real numbers.

Group – A

(Marks : 20)

1. Answer the following multiple choice questions with only one correct option. Choose the correct option and justify : (1+1)×10

(a) The differential equation of the system of circles touching the x-axis at origin is

(i) $\left(x^2-y^2\right)\frac{dy}{dx}-2xy=0$ (ii) $\left(x^2-y^2\right)\frac{dy}{dx}+2xy=0$

(iii) $\left(x^2+y^2\right)\frac{dy}{dx}-2xy=0$ (iv) $\left(x^2+y^2\right)\frac{dy}{dx}+2xy=0$.

(b) Which of the following differential equations is not exact?

(i) $xdy+ydx=0$ (ii) $\sin x\, dy+y\cos x dx=0$

(iii) $\frac{y^2}{x}dx+2y\log_e x\, dy=0$ (iv) $ydx-xdy=0$.

(c) The general solution of the differential equation $\sin px \cos y = \cos px \sin y+p$, where $p=\frac{dy}{dx}$ is

(i) $y=\cos x-\sin^{-1}c$ (ii) $y=cx-\sin^{-1}c$

(iii) $\sin y=cx-\sin^{-1}c$ (iv) $y=c\cos x$.

(d) Which of the following statements is false?

(i) $\sin x$ and $\cos x$ are linearly independent solutions of $\frac{d^2y}{dx^2}+y=0$ on $-\infty<x<\infty$

(ii) e^x and xe^x are linearly independent solutions of $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$ on $-\infty<x<\infty$

(iii) e^x and e^{-x} are linearly independent solutions of $\frac{d^2y}{dx^2}-y=0$ on $-\infty<x<\infty$

(iv) $\sin x$ and $2\sin x$ are linearly independent solutions of $\frac{d^2y}{dx^2}+y=0$ on $-\infty<x<\infty$.

(e) The particular integral of $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=e^{2x}+e^x$ is

(i) $\frac{1}{6}\left(2e^{2x}+3e^x\right)$ (ii) $\frac{1}{6}\left(e^{2x}+e^x\right)$

(iii) $\frac{4e^{2x}+9e^x}{36}$ (iv) $\frac{e^{2x}}{9}+\frac{e^x}{2}$.

(f) Which of the following is correct for the linear differential equation

$(3x+1)x\frac{d^2y}{dx^2}-(x+1)\frac{dy}{dx}+3y=0$?

(i) 0 is an irregular singular point (ii) -1 is an irregular singular point

(iii) -1 is a regular singular point (iv) no irregular singular point.

(g) The domain of definition of the function $f(x, y)=\cos(3x+4y)-\log_e\left(1-x^2-y^2\right)$ is

(i) $D=\left\{(x, y)\in\mathbb{R}^2 : 3x+4y>0\right\}$ (ii) $D=\left\{(x, y)\in\mathbb{R}^2 : 3x+4y<0\right\}$

(iii) $D=\left\{(x, y)\in\mathbb{R}^2 : 0<x^2+y^2<1\right\}$ (iv) $D=\left\{(x, y)\in\mathbb{R}^2 : x^2+y^2<1\right\}$.

(h) The value of $\lim\limits_{(x, y)\to(0,0)}\frac{x\sin\left(x^2+y^2\right)}{x^2+y^2}$

(i) is 1 (ii) is 0 (iii) is -1 (iv) does not exist.

(i) For the function $f(x, y) = x^2 - y^3 - x^2 y + y$, the point $\left(0, \frac{1}{\sqrt{3}}\right)$

(i) is not a critical point (ii) is a saddle point

(iii) is a point of local minimum (iv) is a point of local maximum.

(j) The unit normal to the surface $x^2 + y^2 = z$ at the point (1, 2, 5) is

(i) $2\hat{i} + 4\hat{j} - \hat{k}$ (ii) $-2\hat{i} - 4\hat{j} + \hat{k}$

(iii) $\frac{-2}{\sqrt{21}}\hat{i} - \frac{4}{\sqrt{21}}\hat{j} + \frac{\hat{k}}{\sqrt{21}}$ (iv) $\frac{2}{\sqrt{21}}\hat{i} + \frac{4}{\sqrt{21}}\hat{j} - \frac{\hat{k}}{\sqrt{21}}$.

Group – B

(Marks : 30)

Answer *any six* questions.

2. (a) State the existence and uniqueness theorem for the initial value problem

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0$$

(b) Solve : $\frac{dy}{dx} = e^{x-y}\left(e^x - e^y\right)$ 2+3

3. (a) Solve : $\left(x^2 + y^2\right)dx + \left(x^2 - xy\right)dy = 0$

(b) Solve : $\left(2xy + e^x\right)y\,dx - e^x\,dy = 0$ 3+2

4. Find the value of constant λ such that $\left(2xe^y + 3y^2\right)\frac{dy}{dx} + \left(3x^2 + \lambda e^y\right) = 0$ is exact. Further, for this value of λ, solve the equation. 2+3

5. Reduce the equation $y^2(y - xp) = x^4 p^2$ to Clairaut's form by the substitution $x = \frac{1}{u}$, $y = \frac{1}{v}$ and hence solve it. Also find the singular solution (if it exists). 2+2+1

6. Find the general solution of the following Euler-Cauchy equidimensional equation : 5

$$x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + 2y = x\log_e x$$

7. Solve the following equation by the method of undetermined coefficients : 5

$$\frac{d^2y}{dx^2}-7\frac{dy}{dx}+6y=(x-2)e^x$$

8. Solve by the method of variation of parameters, the equation $\frac{d^2y}{dx^2}+4y=4\tan 2x$ 5

9. Solve for x and y from the system of equations :

$$\frac{dx}{dt}+4x+3y=t$$
$$\frac{dy}{dt}+2x+5y=e^t$$ 5

10. Determine the nature and stability of the critical point (0, 0) of the following system :

$$\frac{dx}{dt}=2x+5y$$
$$\frac{dy}{dt}=x-2y$$

Also draw rough sketch of the corresponding phase portraits. 3+2

11. Find the power series solution of the initial value problem $\frac{d^2y}{dx^2}+\frac{xdy}{dx}+2y=0$, about the point $x=0$. 5

Group – C

(Marks : 15)

Answer *any three* questions.

12. (a) Show that the set $S=\{(x,y)\in\mathbb{R}^2 : 1<x^2+y^2\le 2\}$ is neither open nor closed.

(b) Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$f(x,y)=\begin{cases}\frac{3x^2y}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

Examine if f is continuous at (0, 0). 3+2

13. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^4 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

Examine whether f_x is continuous at (0, 0) and $f_y(0, 0)$ exists. 3+2

14. If $F(p, q, r) = 0$ where $p = v^2 - x^2$, $q = v^2 - y^2$, $r = v^2 - z^2$ and v is a function of x, y, z, show that

$$\frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} + \frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}.$$ 5

15. Find the directional derivative of $x^2 y^2 z^2$ at the point $(1, 1, -1)$ in the direction of the tangent to the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$. 5

16. Examine for existence of maxima or minima of the function $f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$. 5
